

Polymer stars in three dimensions. Three loop results

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We study scaling properties of self avoiding polymer stars and networks of arbitrary given but fixed topology. We use massive field theoretical renormalization group framework to calculate critical exponents governing their universal properties (star exponents). Calculations are performed directly in three dimensions, renormalization group functions are obtained in three loop approximation. Resulting asymptotic series for star exponents are resummed with the help of Padé-Borel and conformal mapping transformation.

I. INTRODUCTION

Many peculiarities in the behavior of polymer chains immersed in a good solvent are well understood qualitatively and described with high precision on the quantitative level due to the application of renormalization group methods [1–3] originating from field theory [4]. This progress was initiated by De Gennes ideas linking the model of the d -dimensional ferromagnet of $O(n)$ -symmetry near its critical temperature $T = T_c$ to self-avoiding walks (SAW) on a lattice of equal dimensionality d [5,6]. In particular correlation functions of the $O(n)$ -symmetric model correspond to those of SAWs in the limit $n = 0$. Let us recall that the critical exponents ν and γ describing the behavior of the correlation length (ξ) and magnetic susceptibility (χ) of the $O(n)$ -model in the vicinity of critical temperature T_c ,

$$\xi \sim \tau^{-\nu},$$

$$\chi \sim \tau^{-\gamma}, \quad \tau = \frac{T - T_c}{T_c}.$$

In polymer theory these exponents correspond to size and configuration number exponents ν and γ describing the average square end-to-end distance R^2

$$R^2 \sim N^{2\nu}, \quad N \gg 1 \tag{1}$$

of a single chain of N monomers and the number \mathcal{Z}_N of possible ways of realization of a SAW of N steps on a given lattice:

$$\mathcal{Z}_N \sim \mu^N N^{\gamma-1}, \quad N \gg 1, \tag{2}$$

where μ is a non-universal fugacity.

These laws have been generalized to describe polymers linked at their endpoints to form polymer networks [7–9,11,10]. It has been shown that for a given network G (2) holds with an exponent $\gamma = \gamma_G$ characteristic of the network. Furthermore a subset of networks called polymer stars consisting of F polymer chains linked at one endpoint gives rise to a basic

series of exponents γ_F . Just as any network may be disassembled to a number of stars cutting polymer chains, its characteristic exponent γ_G is expressed in terms of the exponents γ_F of its constituent stars. Whereas the exponents (1),(2) were intensively studied within the renormalization group approach to high order and by different methods, this is not the case for the star exponents. Their counterparts in the $O(n)$ -symmetric model were introduced as anomalous dimensions of certain composite operators [23] but with no direct physical interpretation. In polymer theory they were introduced independently [7,8] to characterize networks and contacts between chains and later linked to the $n = 0$ limit of the composite operators [13]. With a number of Monte Carlo simulations at hand [14–16] they form an attractive and interesting field for analytic calculations. In this article we present our results for star exponents obtained in the frames of massive field theory renormalization scheme directly in three dimensions. This scheme of calculations as initiated by Parisi [17] avoids the ε -expansion and thus forms an independent check for the validity of previous results [14,15,12,13]. As to our knowledge it is applied here in the context of polymer star properties for the first time.

Let us give a short account of the setup of the article. The next section introduces the model and defines several series of star exponents giving scaling relations between them. In section 3 we calculate the renormalization group functions as series in the coupling to 3rd loop order. Being asymptotic these series are resummed in section 4 by means of Padé-Borel and conformal mapping technique, we give the numerical values of star exponents and discuss them together with results of other approaches. In appendices we show the correspondence between the loop integrals entering the theory and their Feynman diagrams, and give the details of the pseudo- ε -expansion for the star exponents.

II. POLYMER NETWORKS, STAR POLYMERS AND STAR EXPONENTS

Consider a polymer network consisting of long polymer chains tied together at their end-points (fig. 1). Mapping such a network on an equivalent field theory one may prove¹ [12,13] that as in the case of a linear chain [18,19] this theory is multiplicatively renormalizable. The scaling properties of the network are determined by its "star-like" vertices, connecting the extremities of the chains. Let us consider a single star polymer (fig. 2) with F legs of N monomers each. It can be shown that for long chains $N \rightarrow \infty$ the number of self-avoiding configurations will scale according to [10] :

$$\mathcal{Z}_{N,F} \sim \mu^{FN} N^{\gamma_F - 1}, \quad N \gg 1. \quad (3)$$

Formula (3) can be considered as the generalization of (2), μ still being the non-universal connectivity constant and values γ_F give us the first example of star exponents. In the case $F = 1, 2$ one still has the single polymer chain: $\gamma_1 = \gamma_2 = \gamma$ and for $F \geq 3$ γ_F form a set of independent critical exponents. For the two-dimensional case the values of γ_F are known exactly [10,9]:

$$\gamma_F = [68 + 9F(3 - F)]/64.$$

¹ This proof is valid for polymer networks of arbitrary architecture at $d < 4$ and for a large class of molecular weight distributions of the strands.

Exponents γ_F give an example of geometrical exponents associated with special vertices where infinite critical objects are fused together. As mentioned in the introduction there is no direct analogy to critical exponents describing the 2nd order phase transition in $O(n)$ -symmetric model (see, however formulas (9),(20) of this article). Let us note here a similar problem with an *infinite* spectrum of scaling dimensions. Consider a $d = 2$ dimensional percolation cluster and the probability P_F of a pinching point joining F peninsulas of the infinite incipient cluster. Near the percolation threshold p_c this scales like [21,20]:

$$P_F \sim (p - p_c)^{\beta_F},$$

where β_F is another basic set of geometrical critical exponents related to γ_F in $d = 2$. Another example are the so-called contact exponents introduced in [10] (see [7] as well) to describe the mutual behavior of two polymer stars having F and F' legs as function of the distance r of their cores. (fig. 3). Due to the hard-core repulsion the probability of approaching the cores of these two polymers at distance r vanishes as

$$P(r) \sim r^{\theta_{F,F'}}, \quad r \rightarrow 0.$$

Again $\theta_{F,F'}$ form a set of universal contact exponents which can not be expressed in terms of ν and η (1), (2). Let us note that there exist scaling relations between different sets of star exponents. Choosing γ_F exponents as the "basic" ones one gets for the contact exponents $\theta_{F,F'}$ [10]:

$$\nu\theta_{F,F'} - 1 = \gamma_F + \gamma_{F'} - \gamma_{F+F'}.$$

Considering a general network G of arbitrary but fixed topology, made of F chains of equal length N and tied together in vertices (see fig. 1) one has for the asymptotic number of self-avoiding configurations \mathcal{Z}_G :

$$\mathcal{Z}_G \sim \mu^{FN} N^{\gamma_G - 1}, \quad N \gg 1,$$

where the expression for the critical exponent γ_G reads:

$$\gamma_G - 1 + \nu d = \sum_{F \geq 1} n_F \left[(\gamma_F - 1 + \nu d) - \frac{F}{2} (\gamma_2 - 1 + \nu d) \right],$$

here n_F is the number of vertices with F legs.

Thus knowing one set of exponents for single star polymers one can obtain geometrical exponents characterizing any polymer network [8,9,12,13]. So in what follows below we will consider mainly the properties of a single polymer star immersed in a good solvent with arbitrary but fixed number of arms (see fig. 2). We are using the continuous chain model introduced by Edwards (see e.g. [22]). Consider an ensemble of F disconnected branches and put in correspondence to each branch a a path $\mathbf{r}_a(s)$, parametrized by $0 \leq s \leq S_a$, $a = 1, 2, \dots, F$ (S_a being the Gaussian surface of a th branch). One can describe such a system by the following Hamiltonian:

$$\begin{aligned} \frac{1}{k_B T} H_F &= \frac{1}{2} \sum_{a=1}^F \int_0^{S_a} ds \left(\frac{d\mathbf{r}(s)}{ds} \right)^2 + \\ &\quad \frac{u_0}{2} \sum_{a,b=1}^F \int_0^{S_a} ds \int_0^{S_b} ds' \delta^d(\mathbf{r}_a(s) - \mathbf{r}_b(s')). \end{aligned}$$

The partition function is obtained as a functional integral over all possible configurations of polymer system divided by its volume Ω thus dividing out identical configurations just translated in space:

$$\mathcal{Z}\{S_a\} = \frac{1}{\Omega} \int D[\mathbf{r}_1, \dots, \mathbf{r}_F] \exp\left[-\frac{H_F}{k_B T}\right], \quad (4)$$

here the symbol $D[\mathbf{r}_1, \dots, \mathbf{r}_F]$ includes normalization such that $\mathcal{Z}\{S_a\} = 1$ for $u_0 = 0$. To make the exponential of δ -functions and the functional integral well defined in bare theory a cutoff s_0 has to be introduced such that all simultaneous integrals of any variables s and s' are cut off by $|s - s'| > s_0$. With this we can relate the Gaussian surface S of a path to the notion of steps N of a self avoiding walk by

$$N = S/s_0.$$

Introducing into (4) product of δ -functions

$$\prod_{a=2}^F \delta^d(\mathbf{r}^a(0) - \mathbf{r}^1(0))$$

ensuring the “star-like” configuration of a set of F chains (c.f. fig. 2), one gets for the partition function of the polymer star:

$$\mathcal{Z}_F\{S_a\} \equiv \mathcal{Z}_F\{S_1, \dots, S_F\} = \frac{1}{\Omega} \int D[\mathbf{r}_1, \dots, \mathbf{r}_F] \exp\left[-\frac{H_F}{k_B T}\right] \times \prod_{a=2}^F \delta^d(\mathbf{r}^a(0) - \mathbf{r}^1(0)).$$

Besides the partition functions one can define correlation functions of an F -arm star. For the core-endpoint correlation function one has [8,9,12,13]:

$$\hat{P}_F(\mathbf{r}_0; \mathbf{r}_1, \dots, \mathbf{r}_F; S_1, \dots, S_F) = \left\langle \prod_{a=1}^F \delta^d(\mathbf{r}^a(0) - \mathbf{r}_0) \delta^d(\mathbf{r}^a(S_a) - \mathbf{r}_a) \right\rangle, \quad (5)$$

here \mathbf{r}_0 is the coordinate of the core, $\mathbf{r}_1, \dots, \mathbf{r}_F$ being coordinates of chain endpoints, S_1, \dots, S_F being their Gaussian surfaces. The averaging in (5) means:

$$\langle \dots \rangle = \frac{\frac{1}{\Omega} \int D[\mathbf{r}_1, \dots, \mathbf{r}_F] \exp\left[-\frac{H_F}{k_B T}\right] (\dots)}{\mathcal{Z}_F\{S_a\}}.$$

For the Green functions one has:

$$(2\pi)^d \delta^d[\mathbf{p}_0 + \sum_{a=1}^F \mathbf{p}_a] G_F(\mathbf{p}_0; \mathbf{p}_1, \dots, \mathbf{p}_F; S_1, \dots, S_F) = \mathcal{Z}_F(S_1, \dots, S_F) \int \prod_{a=0}^F [d^d \mathbf{r}_a \exp[i\mathbf{p}_a \mathbf{r}_a] \hat{P}_F(\mathbf{r}_0; \mathbf{r}_1, \dots, \mathbf{r}_F; S_1, \dots, S_F). \quad (6)$$

The mapping to field theory is performed by a Laplace transformation from the Gaussian surfaces S_a to conjugated chemical potential variables ("mass variables") μ_a :

$$\tilde{\mathcal{Z}}\{\mu_a\} = \int \prod_{b=1}^F dS_b \exp[-\mu_b S_b] \mathcal{Z}_F\{S_a\}. \quad (7)$$

As a result an ensemble of F polymer chains is described by a field theory with the Lagrangean $\mathcal{L}\{\phi_a, \mu_a\}$ involving a separate field $\vec{\phi}_a \equiv \vec{\phi}_a(\mathbf{r})$ for each chain:

$$\mathcal{L}\{\phi_b, \mu_b\} = \frac{1}{2} \sum_{a=1}^F \int d^d \mathbf{r} (\mu_a |\vec{\phi}_a|^2 + \frac{1}{2} |\nabla \vec{\phi}_a|^2) + \frac{u_0}{8} \sum_{a,a'=1}^F \int d^d \mathbf{r} |\vec{\phi}_a|^2 |\vec{\phi}_{a'}|^2, \quad (8)$$

here $\vec{\phi}_a$ is an n -component vector field $\vec{\phi}_a = (\phi^1, \dots, \phi^n)$, $|\vec{\phi}_a|^2 = \sum_{\alpha=1}^n (\phi_a^\alpha)^2$. However, the perturbation expansion of field theory (8) at arbitrary value of n results in particular in some diagrams, which do not appear in polymer theory. As it is well known [5,6] such diagrams (involving closed loops of propagator lines connected to the remainder of the diagram by interaction lines only) can be suppressed by taking the limit $n = 0$.

Returning now to the partition function of a single polymer star $\mathcal{Z}_F\{S_b\}$ and using the Laplace transformation (7) we can represent its Laplace transform $\tilde{\mathcal{Z}}_F\{\mu_b\}$ in the following form:

$$\begin{aligned} \tilde{\mathcal{Z}}_F\{\mu_b\} = & \frac{1}{\Omega} \int d^d \mathbf{r}_0 \prod_{a=1}^F d^d \mathbf{r}_a \int D[\phi_a(r)] \times \\ & \prod_{a=1}^F \phi_a(r_0) \phi_a(r_a) \exp[-\mathcal{L}\{\phi_b, \mu_b\}] \big|_{n=0}. \end{aligned}$$

In field theory the star vertex is related to the local composite operator $\prod_{a=1}^F \phi_a$ of F distinct zero component fields. Formally this is the $n = 0$ limit of an operator known in n -component field theory [23]:

$$\sum_{\alpha_1, \dots, \alpha_F=1}^n N^{\alpha_1, \dots, \alpha_F} \phi^{\alpha_1}(r) \dots \phi^{\alpha_F}(r), \quad (9)$$

where $N^{\alpha_1, \dots, \alpha_F}$ is a zero trace symmetric $SO(n)$ tensor:

$$\sum_{\alpha=1}^n N^{\alpha, \alpha, \alpha_3, \dots, \alpha_F} = 0.$$

Now it can be checked diagrammatically that Green functions involving an operator with $N^{\alpha_1, \dots, \alpha_F}$ symmetry coupled to F external fields $\phi_a^\alpha(r_a)$ in the limit $n = 0$ coincide with Green functions (6) of the F -arm star polymer. Thus in order to find star exponents in the field-theoretical renormalization group program one may consider the problem of renormalization of composite operators (9). This will be the subject of the subsequent section.

III. RENORMALIZATION GROUP FUNCTIONS IN THREE LOOP APPROXIMATION

As was shown in the previous section considering the behavior of the star-like vertices in polymer theory one faces the problem of renormalization of a field theory containing two couplings one being of $O(n)$ symmetry and described by tensor $S_{\alpha_1, \dots, \alpha_4}$, the other being of traceless $SO(n)$ symmetry ($N^{\alpha_1, \dots, \alpha_F}$):

$$\mathcal{L}\{\phi_b, \mu_b\} = \frac{1}{2} \int d^d \mathbf{r} [(\mu_0^2 |\vec{\phi}|^2 + |\nabla \vec{\phi}|^2) + \frac{u_0}{4!} S_{\alpha_1, \dots, \alpha_4} \phi^{\alpha_1} \dots \phi^{\alpha_4} + \frac{v_0}{4!} N^{\alpha_1, \dots, \alpha_F} \phi^{\alpha_1} \dots \phi^{\alpha_F}]. \quad (10)$$

Let us point out that the problem of composite field renormalization in the case of traceless symmetry we are going to tackle here was considered in the frames of ε -expansion in [23] in the field-theory context, in polymer context ε -expansion results up to ε^3 -order were analysed in [12,13]. We will return to these results later. The distinct feature of our study is that we are going to apply here equations of the massive field theory. As is well known in order to analyze the expressions appearing in such an approach one can either apply ε -expansion (and thus re-derive the results of [12,13,23]) or as it was proposed by Parisi [17] consider renormalization group functions directly in the dimension of space of interest (here we are interested in the three-dimensional case). So performing calculations in the spirit of [17] enables us to proceed directly in three dimensions.

The critical properties of the field theory (10) can be extracted from the coefficients $\beta_u, \beta_v, \gamma_\phi, \gamma_{\phi^2}$ of the Callan-Symanzik equation for the renormalized N -point vertex function $\Gamma_R^{(N)}$ (see e.g. [3,1,2]). As far as we are interested in the renormalization group functions and their derivatives over the coupling constants at the Heisenberg fixed point determined by the stable solution of $\beta_u(u^* \neq 0, v^* = 0) = 0$ we need to calculate vertex functions of the symmetric interaction $S_{\alpha_1, \dots, \alpha_4} \phi^{\alpha_1} \dots \phi^{\alpha_4}$ $\Gamma^{(2)}, \Gamma^{(4)}$ together with F -point vertex function $\Gamma^{(F)}$ with only one $N^{\alpha_1, \dots, \alpha_F} \phi^{\alpha_1} \dots \phi^{\alpha_F}$ insertion (the other terms will either contain some trace of tensor $N^{\alpha_1, \dots, \alpha_F}$ or will be proportional to some power of v and will disappear in the Heisenberg fixed point). The graphs for $\Gamma^{(F)}$ are obtained from the usual four-point graphs (see fig. 5) by replacing each four-point vertex in turn by $v N^{\alpha_1, \dots, \alpha_F}$. In three loop approximation which we are going to consider here there appear two more graphs in $\Gamma^{(F)}$ which can not be produced in this manner (they are shown in fig. 4). Finally the expressions for β - and γ -functions read ²:

$$\beta_u = -(4-d)u \left[1 - u + \beta_u^{2LA} u^2 + \beta_u^{3LA} u^3 + \dots \right], \quad (11)$$

²We changed the scale for the renormalized couplings $u = u'(n+8)D/6$, $v = v'D/6$ and beta functions $\beta_u(u) = 6\beta_{u'}(u')/[(n+8)D]$, $\beta_v(u) = 6\beta_{v'}(v')/D$, (D being the one-loop integral: $D = \frac{1}{(2\pi)^d} \int \frac{d\vec{k}}{(k^2+1)^2}$, u', v' being the renormalized couplings corresponding to the bare couplings u_0, v_0), to define a convenient numerical scale in which the first two coefficients of $\beta_u(u)$ are -1 and 1 at $d = 3$.

$$\beta_v = -(4-d)v\left[\frac{\delta_F}{(4-d)} - \frac{F(F-1)}{n+8}(u + \beta_v^{2LA}u^2 + \beta_v^{3LA}u^3 + \dots)\right], \quad (12)$$

$$\gamma_\phi = -(4-d)\frac{2(n+2)}{(n+8)^2}u^2[2i_2 + (4i_2 - 3i_8)u + \dots], \quad (13)$$

$$\tilde{\gamma}_{\phi^2} = (4-d)\frac{n+2}{n+8}u\left[1 + \gamma^{2LA}u + \gamma^{3LA}u^2 + \dots\right], \quad (14)$$

here δ_F is the engineering dimension of the coupling v :

$$\delta_F = F + d - \frac{Fd}{2}$$

and expressions for two-loop $(\beta_u^{(2LA)}, \beta_v^{(2LA)}, \gamma^{(2LA)})$ and three-loop $(\beta_u^{(3LA)}, \beta_v^{(3LA)}, \gamma^{(3LA)})$ contributions read:

$$\begin{aligned} \beta_u^{(2LA)} &= \frac{8}{(n+8)^2}[(5n+22)(i_1 - \frac{1}{2}) + i_2(n+2)], \\ \beta_v^{(3LA)} &= -(2i_1(n+4F-2) + 2i_2(n+2)/(F-1) + (-n-4F+2))/(n+8), \\ \gamma^{(2LA)} &= \frac{6}{(n+8)}(1 - 2i_1), \\ \beta_u^{(3LA)} &= \frac{1}{(n+8)^2} \sum_{j=0}^8 b_u^j i_j, \end{aligned} \quad (15)$$

$$\beta_v^{(3LA)} = \frac{1}{(n+8)^2} \sum_{j=0}^8 b_v^j i_j, \quad (16)$$

$$\begin{aligned} \gamma^{(3LA)} &= \frac{1}{(n+8)^2}[10(n+8) - (44n+280)i_1 + \\ &\quad + (8-3d)(n+2)i_2 + 12(n+2)i_3 + \\ &\quad + 24(n+8)i_4 + 6(n+8)i_5 + 18(n+2)i_6]. \end{aligned} \quad (17)$$

In (15), (16) $i_0 \equiv 1$. And for the coefficients b_u^j, b_v^j one has:

$$\begin{aligned} b_u^0 &= -8(4n^2 + 61n + 178), \\ b_u^1 &= 4(31n^2 + 430n + 1240), \\ b_u^2 &= (3dn^2 + 30dn + 48d + 8n^2 + 80n + 128), \\ b_u^3 &= -12(n^2 + 10n + 16), \\ b_u^4 &= -48(n^2 + 20n + 60), \\ b_u^5 &= -24(2n^2 + 21n + 58), \\ b_u^6 &= -6(3n^2 + 22n + 56), \\ b_u^7 &= -24(5n + 22), \\ b_u^8 &= -12(n^2 + 10n + 16); \end{aligned}$$

$$\begin{aligned}
b_v^0 &= (n^2 + 8nF + 6n + 20F^2 - 28F + 56), \\
b_v^1 &= -4(n^2 + 7nF + 5n + 18F^2 - 28F + 54), \\
b_v^2 &= -(3dnF - 3dn + 6dF - 6d + 4n^2 - \\
&\quad 8nF + 48n - 16F + 80)/(F - 1), \\
b_v^3 &= 12(n + 2), \\
b_v^4 &= 12(nF + 2n + 4F^2 - 10F + 20), \\
b_v^5 &= 3(n^2 + 4nF - 2n + 4F^2 + 12F - 24), \\
b_v^6 &= 6(n + 2F^2 - 10F + 18), \\
b_v^7 &= 4(nF - 2n + 14F - 28), \\
b_v^8 &= 3(n^2 + 10n + 16)/(F - 1).
\end{aligned} \tag{18}$$

here $i_1 - i_8$ are the integrals originating from the corresponding two- and three-loop Feynman graphs. Their numerical values at $d = 3$ are as follows [24,25]:

$$\begin{aligned}
i_1 &= 2/3; \quad i_2 = -2/27; \quad i_3 = -.0376820725; \\
i_4 &= .3835760966; \quad i_5 = .5194312413; \quad i_6 = 1/2; \\
i_7 &= .1739006107; \quad i_8 = -.0946514319.
\end{aligned} \tag{19}$$

Correspondence between $i_1 - i_8$ and the appropriate Feynman graphs is given in the Appendix A. For the additional graphs appearing in $\Gamma^{(F)}$ one finds the corresponding normalized numerical values to be equal: 1 (fig. 4a) and i_1 (fig.4b).

In the case $d = 3$ expression for β_u and gamma-functions coincide with three-loop parts of appropriate expressions obtained in [26] and at arbitrary value of d they were given in [27].

Note that expressions (11) - (14) are written for the arbitrary number of field components n and thus contain more information that is necessary for the polymer case $n = 0$. This case was considered by Wallace and Zia [23] in three loop ε -expansion motivated by the question of relevance of the v -coupling in the renormalization group sense. These authors introduced a series of critical exponents α_F : the value of the critical exponent α_F is connected with the anomalous dimension x'_F of v at the $O(n)$ -symmetrical fixed point:

$$\begin{aligned}
(2 - \eta)\alpha_F &= x'_F - \frac{F}{2}\eta, \\
x'_F &= \frac{\partial\beta_v(u, v)}{\partial v} \Big|_{u=u^*, v=0} - \delta_F.
\end{aligned} \tag{20}$$

The relation to the star exponents is given in the limit $n = 0$ by

$$\gamma_F - 1 = -\nu(2 - \eta)\alpha_F + [\nu(2 - \eta) - 1]F. \tag{21}$$

Thus the expressions (11)-(14) may be used to study the stability of the $O(n)$ -symmetric fixed point to the perturbation introduced by the traceless coupling.

IV. STAR EXPONENTS IN THREE DIMENSIONS

We now proceed with the calculation of the star exponents. Combining (11)-(14) with (21) one obtains the following expression for the function $\gamma_F(u)$ ($\gamma_F = \gamma_F(u^*)$)³:

$$\gamma_F = 1 + (4 - d)F[\gamma_F^{1LA}u + \gamma_F^{2LA}u^2 + \gamma_F^{3LA}u^3], \quad (22)$$

$$\gamma_F^{1LA} = \frac{n - F + 3}{2(n + 8)}, \quad (23)$$

$$\begin{aligned} \gamma_F^{2LA} = \frac{-1}{(n + 8)^2} [& (-nF + 7n - 4F^2 + 6F + 10)i_1 + \\ & (dn^2 - dnF + 5dn - 2dF + 6d - 4n^2 + 6nF - \\ & 34n + 8F^2 - 4F - 44)/4], \end{aligned} \quad (24)$$

$$\gamma_F^{3LA} = \frac{1}{(n + 8)^3} \sum_{j=0}^8 \gamma_F^j i_j, \quad (25)$$

$$\begin{aligned} \gamma_F^0 = \frac{-1}{8} (& -d^2n^3 + d^2n^2F - \\ & 7d^2n^2 + 4d^2nF - 16d^2n + 4d^2F - 12d^2 + \\ & 8dn^3 - 10dn^2F + 82d * n^2 - 8dnF^2 - \\ & 36dnF + 236dn - 16dF^2 - 32dF + 208d - 16n^3 + \\ & 28n^2F - 260n^2 + 64nF^2 + 72nF - 1112n + \\ & 80F^3 - 128F^2 + 400F - 1504), \end{aligned}$$

$$\begin{aligned} \gamma_F^1 = \frac{-1}{2} (& dn^2F - 13dn^2 + \\ & 4dnF^2 + 2dnF - 54dn + 8dF^2 - 56d - 8n^2F + 100n^2 - \\ & 44nF^2 + 604n - 72F^3 + 152F^2 - 328F + 1000), \end{aligned}$$

$$\gamma_F^2 = \frac{-d}{2} (n^2 - nF + 5n - 2F + 6),$$

$$\gamma_F^3 = -6(-n^2 + nF - 5n + 2F - 6),$$

$$\begin{aligned} \gamma_F^4 = & -6(-2n^2 + nF^2 + nF - 22n + 4F^3 - \\ & -14F^2 + 30F - 52), \end{aligned}$$

$$\begin{aligned} \gamma_F^5 = \frac{-3}{2} (& n^2F - 3n^2 + 4nF^2 - 6nF - \\ & 18n + 4F^3 + 8F^2 - 36F - 8), \end{aligned}$$

$$\begin{aligned} \gamma_F^6 = & -3(-3n^2 + nF - 13n + 2F^3 - \\ & 12F^2 + 28F - 30), \end{aligned}$$

$$\gamma_F^7 = -2(nF^2 - 3nF + 2n + 14F^2 - 42F + 28).$$

³ Note scaling relation (30) allowing one to find connection between $\gamma_F(u)$ and functions (11) - (14) on the basis of (20). Critical exponent η and the combination $2 - \nu^{-1} - \eta$ are given by the fixed point values of γ_ϕ , $\tilde{\gamma}_{\phi^2}$ (13),(14)

Calculating star exponents γ_F (22) one should solve first the fixed point equation:

$$\beta_u(u^*) = 0 \quad (26)$$

and then calculate series (22) at $u = u^*$ resulting in the value of star exponent γ_F :

$$\gamma_F = \gamma_F(u = u^*). \quad (27)$$

In order to be self-consistent substituting result of (26) into the series for γ_F we introduce here pseudo- ε -expansion (See Appendix B). The resulting series for γ_F (B.2) (as well as (11) - (14)) is known to be asymptotic and the appropriate resummation procedure is to be applied to obtain from it reliable information.

The results given below were obtained by applying two different resummation techniques to the series for $\gamma_F(\tau)$ (B.2) as a function of “pseudo- ε ” parameter τ ⁴:

$$\gamma_F(\tau) = \sum_j \tilde{\gamma}_F^j \tau^j. \quad (28)$$

First we tried simple Padé-Borel resummation using [2/1] Padé-approximant $\gamma_F^P(\tau)$ for analytical continuation of the Borel transform $\gamma_F^B(\tau t)$ of $\gamma_F(\tau)$:

$$\gamma_F^B(\tau t) = \sum_j \frac{\tilde{\gamma}_F^j}{j!} (\tau t)^j$$

and writing for the resummed function:

$$\gamma_F^{res}(\tau) = \int_0^\infty dt e^{-t} \gamma_F^P(\tau t).$$

Then we applied resummation procedure based on the conformal mapping transformation, mapping the domain of analyticity of $\gamma_F^B(\tau)$ containing the real positive axis onto a circle centered at the origin. Here we introduced a fit parameter (b) considering instead of Borel-transform the Borel-Leroy transform $\gamma_F^{B-L}(\tau)$ of $\gamma_F(\tau)$ defined by:

$$\gamma_F^{B-L}(\tau) = \sum_j \frac{\tilde{\gamma}_F^j}{\Gamma(j+b+1)} (\tau)^j, \quad (29)$$

then

$$\gamma_F(\tau) = \int_0^\infty dt t^b e^{-t} \gamma_F^{B-L}(\tau t).$$

Assuming that the singularity of $\gamma_F^{B-L}(\tau)$ closest to the origin is located at the point $(-1/a)$ we mapped the τ plane onto a circle with a mapping leaving the origin invariant:

$$w = \frac{(1+a\tau)^{1/2} - 1}{(1+a\tau)^{1/2} + 1},$$

⁴ Let us recall that star exponent γ_F equals $\gamma_F = \gamma_F(\tau = 1)$.

$$\tau = \frac{4}{a} \frac{w}{(1-w)^2}$$

and thus obtained an expression for $\gamma_F^{B-L}(\tau)$ convergent in the whole cut plane and, as a result, the expression for the resummed function γ_F^{res} . While doing this in order to weaken a possible singularity on w -plane we multiplied the corresponding expression by $(1-w)^\alpha$ and thus introduced one more parameter α . In the resummation procedure the value of a (the location of the closest singularity in pseudo- ε -expansion) was taken from the known large-order behavior of ε -expansion series for critical exponents [28], while α was chosen as a fit parameter defined by the condition of minimal difference between resummed two- and three-loop results. The resummation procedure was quite insensitive to the parameter b introduced by the Borel-Leroy transformation (29).

The results obtained are given in table 4. First we give the values of the exponent γ_F obtained by Padé-Borel resummation: the second column contains the value of γ_F obtained directly from resummation of the series (28), while the third column gives γ_F obtained on the base of the resummed series for the exponent x'_F (namely such a way of calculation of γ_F was chosen in [13,12] using the ε -expansion method). In this case the numerical value of γ_F was obtained on the base of the numerical value of x'_F via scaling relation:

$$\gamma_F = 1 - \nu x'_F + [\nu(2 - \eta/2) - 1]F \quad (30)$$

substituting well-known values of the exponents $\nu(d=3) = 0.588$, $\eta(d=3) = 0.027$. Columns 4 and 5 give γ_F obtained by the resummation based on conformal mapping technique: resummation of the series $\gamma_F(\tau)$ (fourth column) and resummation of the series $x'_F(\tau)$ (fifth column). The next columns give the results obtained by the ε^3 -expansion based on: simple Padé approximation (the 6th column) and Padé-Borel analysis neglecting or exploiting exact results for $d=2$ (the 7th and 8th columns respectively) [13,12]. The last column contains Monte-Carlo data [14,15]. For low number of arms $F \leq 5$ the results of the different approaches agree reasonably well and are also close to the values obtained by MC simulation. We have used two different renormalization schemes as well as different procedures for the resummation of the resulting asymptotic series. Table 4 gives thus a test for the stability of the results under changes of the calculational scheme. Obviously for higher number $F > 5$ of arms coincidence of the results is no longer good. The main reason for this is that calculating the exponents combinatorial factors lead to an expansion in $F\varepsilon$ for the ε expansion and of Fg when directly expanding in a renormalized coupling g . For such large values of the expansion parameter even resummation of the series fails. For large numbers of arms other approaches to the theory of polymer stars like a self consistent field approximation might be more useful. We conclude that the Parisi method of massive renormalization in fixed dimension as it is widely applied in the theory of critical phenomena used together with an appropriate resummation scheme is a powerful tool also for the calculation of exponents in polymer theory in the present case leading to a good test of previous results and methods.

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APPENDIX A: APPENDIX

Here we give the correspondence between loop integrals entering renormalization group functions ((11) - (14) and below) and appropriate Feynman graphs for one-particle irreducible vertex functions. Fig. 5 give graphs up to the three loop order for the functions $\Gamma^{(2)}$, $\Gamma^{(4)}$ (we keep labeling of [24]). Two additional graphs appearing in $\Gamma^{(F)}$ are given by fig.4a,b. In table 4 we show the correspondence between numerical values of the loop integrals and appropriate Feynman graphs.

APPENDIX B: APPENDIX

In the frames of pseudo- ε -expansion ⁵ an auxiliary parameter (let us denote it by τ) is introduced to keep track of the orders of perturbation theory. The corresponding expression for function β_u (11) reads:

$$\beta_u(u, \tau) = -\tau u + \sum_{j=2}^{\infty} \beta_u^{(j-1)LA} u^j,$$

β_u^{jLA} being j -loop contributions. The equality holds:

$$\beta_u(u, \tau = 1) = \beta_u(u).$$

Now one can obtain the value of the fixed point u^* as a series in τ ⁶:

$$u^*(\tau) = \sum_j u^{*,(j)} \tau^j. \quad (\text{B.1})$$

This series can be substituted into (22) resulting in the expression for $\gamma_F(\tau)$ being series in powers of “pseudo- ε ” parameter τ . and the final formula for critical exponent γ_F read:

$$\gamma_F = \gamma_F^{res}(u^*, \tau = 1).$$

Where γ_F^{res} means the resummed (with respect to t) series $\gamma_F(\tau)$. Performing expansion (B.1) on the base of the expression for the β -function (11) and substituting the series for the Heisenberg fixed point $u^* \neq 0$, $v^* = 0$ into (22) one finally obtains the following expansion of γ_F in powers of the “pseudo- ε ” parameter τ :

$$\gamma_F(t) = \sum_{j=0}^3 \tilde{\gamma}_F^j t^j. \quad (\text{B.2})$$

⁵ First introduced by B.Nickel: see Ref.19 in [29].

⁶ In fact powers of τ in a certain term of perturbation theory correspond to the number of loops in the loop integrals and this enables one to separate contributions from different orders of the perturbation theory while substituting results of (26) into (27).

The coefficients $\tilde{\gamma}_F^j$ for $j = 0 - 3$ read:

$$\begin{aligned}
\tilde{\gamma}_F^0 &= 1, \\
\tilde{\gamma}_F^1 &= (4-d)F\gamma_F^{(1LA)}, \\
\tilde{\gamma}_F^2 &= (4-d)F\gamma_F^{2LA} + \frac{8(4-d)F\gamma_F^{1LA}}{(n+8)^2} \times \\
&\quad ((5n+22)(i_1 - \frac{1}{2}) + (n+2)i_2), \\
\tilde{\gamma}_F^3 &= (4-d)F\gamma_F^{3LA} + \frac{16(4-d)F\gamma_F^{2LA}}{(n+8)^2} [(5n+22)(i_1 - \frac{1}{2}) + \\
&\quad (n+2)i_2] + (4-d)F\gamma_F^{1LA} [\frac{128}{(n+8)^4} ((5n+22)(i_1 - \frac{1}{2}) + \\
&\quad +(n+2)i_2)^2 + \frac{1}{(n+8)^3} (-32n^2 - 488n - 1424 + 4(31n^2 + \\
&\quad 430n + 1240)i_1 + (3d+8)(n+2)(n+8)i_2 - \\
&\quad 12(n+2)(n+8)i_3 - 48(n^2 + 20n + 60)i_4 - \\
&\quad 24(2n^2 + 21n + 58)i_5 - 6(3n^2 + 22n + 56)i_6 - \\
&\quad 24(5n+22)i_7 - 12(n+2)(n+8)i_8)].
\end{aligned}$$

expressions for γ_F^{1LA} , γ_F^{2LA} , γ_F^{3LA} are defined in (23) - (25).

TABLE I. Values of the star exponent γ_F obtained in three-dimensional theory (columns 2,3,4,5) in comparison with the results of ε -expansion (columns 6,7,8) and Monte-Carlo simulations (column 9). See the text for a full description.

F	$d = 3$ Padé-Borel		$d = 3$ conf.mapping		ε -expansion [13,12]			MC [14,15]
3	1.06	1.05	1.06	1.06	1.05	1.05	1.07	1.09
4	0.86	0.86	.86	.83	0.84	0.83	0.85	0.88
5	0.61	0.61	.58	.56	0.53	0.52	0.55	0.57
6	0.32	0.32	.24	.22	0.14	0.18		0.16
7	-0.02	-0.01	-.17	-.17	-0.33	-0.20		
8	-0.40	-0.36	-.63	-.62	-0.88	-0.60		(-0.99, -0.30)
9	-0.80	-0.72	-1.14	-1.11	-1.51	-1.01		

TABLE II. Normalized values of the loop integrals (for graphs 2-M1 - 5-S3 for two-point function $\Gamma^{(2)}(k)$ the corresponding derivative $\partial/\partial k^2|_{k^2=0}$ is given.

Graph	Integral value	Graph	Integral value	Graph	Integral value
2-U2	1	8-U4	i_4	fig.4b	i_1
3-U3	1	9-U4	i_5	2-M1	0
4-U3	i_1	10-U	i_6	3-S2	i_2
5-U4	1	11-U4	i_5	4-M3	0
6-U4	i_1	12-U4	i_7	5-S3	i_8
7-U4	i_3	fig.4a	1		

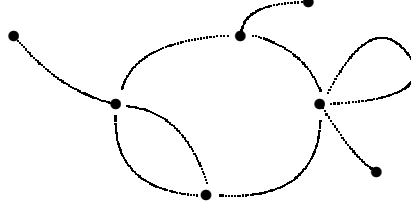


FIG. 1. A polymer network G . It is characterized by the numbers n_F of F -leg vertices. Here $n_1 = 3$, $n_3 = 2$, $n_4 = 1$, $n_5 = 1$.

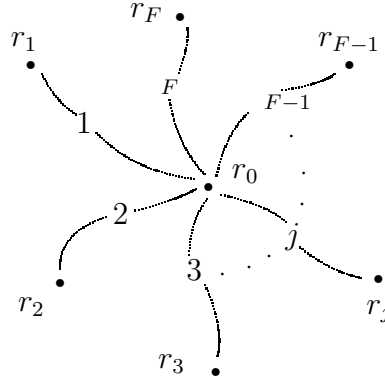


FIG. 2. Star polymer.

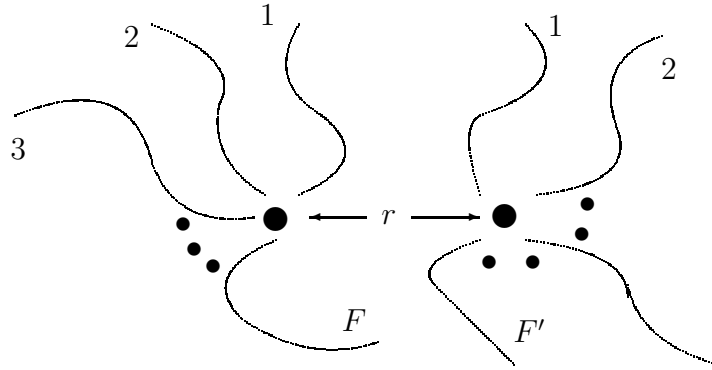


FIG. 3. Two star polymers of functionalities F and F' at a distance r .

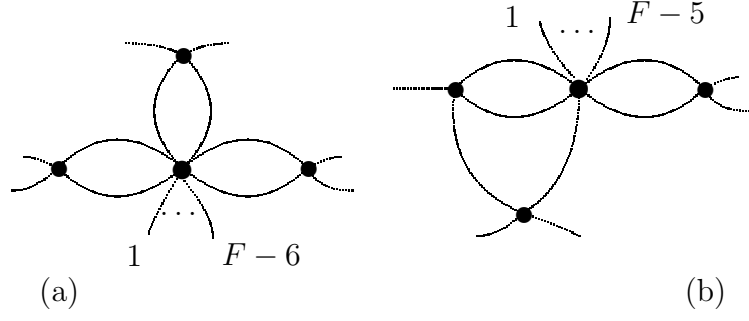


FIG. 4. Additional graphs appearing in the function Γ^F in three-loop approximation.

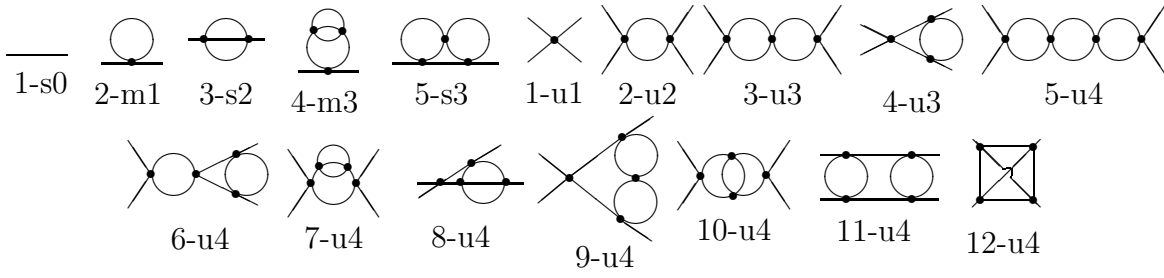


FIG. 5. Graphs of functions $\Gamma^{(2)}$, $\Gamma^{(4)}$ in three-loop approximation.